An EOQ Model for Deteriorating Items Quadratic Demand and Shortages

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Scope and Purpose: The scope of this model includes the application of inventory management. It has been empirically observed that the life expectancy of many items can be expressed in terms of Weibull distribution. This empirical observation has prompted researchers to represent the time to deterioration of a product by a Weibull distribution. It is also observed that demand of a consumer product usually varies with time and hence, the demand rate should be taken as time-dependent. Our purpose is to focus on a mathematical model on inventory production system considering all these factors. Therefore, we develop an economic inventory model for deteriorating items with quadratic demand and shortages in inventory.

Abstract

This paper presents an inventory model for deteriorating items with quadratic demand, instantaneous supply and shortages in inventory. A two-parameter Weibull distribution is taken to represent the time to deterioration. The theory for finding the optimal solution of the problem is developed. A numerical example is taken to illustrate the solution procedure. Keywords: EOQ, time-quadratic demand, shortages, deterioration.

1. Introduction:

In inventory problems, deterioration is defined as damage, decay, spoilage, evaporation, obsolescence, loss of utility or loss of marginal value of goods that results in decrease the usefulness of the original one. Deterioration should not be neglected in inventory problems for the items like foodstuff, chemicals,
pharmaceuticals, electronic goods, radioactive substances, etc. Emmons’s (1968) models with two-parameter Weibull distribution deterioration were discussed by Covert and Philip (1973), Philip (1974), Giri et al. (2003), Ghosh and Chaudhuri (2004) etc. whereas Chakrabarty, Giri and Chaudhuri (1998) and other researchers used three-parameter Weibull distribution deterioration in their inventory models. Giri et al. (1999), Sana et al. (2004), Sana and Chaudhuri (2004a), etc., developed inventory models in this direction. Misra (1975) developed an EOQ model with a Weibull deterioration rate for perishable product where backordering is not allowed. These investigations were followed by several researchers like Deb and Chaudhuri (1986), Goswami and Chaudhuri (1991), Giri et al. (1996) etc. where a time-proportional deterioration rate is considered.

It has been empirically observed that the failure and life expectancy of many items can be expressed in terms of Weibull distribution. This empirical observation has encouraged researchers to represent the time of deterioration of a product by Weibull distribution. Ghare and Schrader’s (1963) model was extended by Covert and Philip (1973) and obtained an EOQ model with a variable rate of deterioration by assuming a two-parameter Weibull distribution. Later, many researchers like Tadikamalla (1978), Chakrabarty et al. (1998), Mukhopadhyay et al. (2004, 2005) developed economic order quantity models. Therefore, the rate of deterioration is treated as time varying function in realistic models. Begum et al. (2010) develop an EOQ model for varying deteriorating items with Weibull distribution deterioration and price-dependent demand. They assume that the demand and deterioration rates are continuous and differentiable function of price and time.

Demand plays a key role in modeling of deteriorating inventory, researchers have recognized and studied the variations (or their combinations) of demand from the viewpoint of real life situations. Demand may be constant, time-varying, stock-dependent, price-dependent, etc. The constant demand is valid, only when the phase of the product life cycle is matured and also for finite periods of time. Wagner and Whitin (1958) discussed the discrete case of the dynamic version of EOQ. The classical no-shortage inventory policy for linear trend in demand was discussed by Donaldson (1977). EOQ models for deteriorating items with trended demand were considered by Bahari-Kashani (1989), Goswami and Chaudhuri (1991, 1992), Xu and Wang (1990), Kim (1995), Jalan et al. (1996), Jalan and Chaudhuri (1999), Lin et al. (2000), etc. Many research articles by Silver (1979), Henery (1979), McDonald (1979),
Dave and Patel (1981), Sachan (1984), Deb and Chaudhuri (1986), Murdeshwar (1988), Hariga (1993), etc. analyzed linear time-varying demand. Later, Ghosh and Chaudhuri (2004, 2006), Khanra and Chaudhuri (2003), etc. established their models with quadratic time-varying demand. In the present paper, we reconsider the model of Covert and Philip (1973) and extend it to include a time-quadratic demand rate and shortages in inventory. A two-parameter Weibull distribution is considered to represent the time to deterioration. The theory for finding the optimal solution of the problem is developed. A numerical example is taken to illustrate the solution procedure. Sensitivity of the optimal solution with respect to changes in different parameter values is examined.

2. Assumptions and Notations of the Model

The model is developed using the following assumptions:

(a) The deterministic demand rate $D(t)$ varies in quadratic with time.

\[ D = a + bt + ct^2 \]

where $a$, $b$ and $c$ are constants. Here '$a'$ is initial rate of demand, '$b'$ is the rate at which the demand rate increases.

(b) Lead time is zero.

(c) The replenishment is instantaneous.

(d) Shortages are allowed.

(e) The holding cost, ordering cost, shortage cost and unit cost remain constant over time.

(f) The distribution of the time to deterioration follows a two-parameter Weibull distribution and the deteriorated units are not replaced during a given cycle.

To develop the mathematical model of the inventory replenishment, the notations adopted in this paper is listed below:

\[ K = \text{a constant value } (0 < K < 1) \]

\[ c_1 = \text{carrying cost per unit per unit time} \]

\[ c_2 = \text{shortage cost per unit per unit time.} \]

\[ c_3 = \text{ordering cost per order} \]

\[ c_4 = \text{cost of a unit} \]
length of the inventory cycle.

\[ D_t = \text{demand rate at any instant } 't', \text{ i.e. } D(t) = a + bt + ct^2 \text{ where } a, b, c \text{ are positive constants.} \]

\[ \theta(t) = \text{instantaneous rate of deterioration of the inventory is followed by a two-parameter Weibull distribution, i.e. } \theta(t) = \alpha \beta t^{\beta-1}, \alpha, \beta > 0. \]

\[ a = \text{scale parameter, } a > 0. \]

\[ \beta = \text{shape parameter, } \beta > 0. \]

### 3. Formulation of the Model

Let \( Q(t) \) be the instantaneous inventory level at any time \( t \geq 0 \). The instantaneous state of \( Q(t) \) at any time \( t \) is described by the differential equation

\[
\frac{dQ(t)}{dt} + \theta(t)Q(t) = -(a + bt + ct^2), \quad 0 \leq t \leq t_1
\]

\[
\frac{dQ(t)}{dt} = -(a + bt + ct^2), \quad t_1 \leq t \leq T
\]

Taking \( \theta(t) = \alpha \beta t^{\beta-1} \), \( \alpha, \beta, t > 0 \)

Equation (1) becomes

\[
\frac{dQ}{dt} + \alpha \beta t^{\beta-1}Q = -(a + bt + ct^2), \quad 0 \leq t \leq t_1
\]

The solution of equation (4) yields

\[
e^{\alpha \beta \xi} Q(\xi) = q_0 - \int_0^\xi ((a + bt + ct^2)) e^{\alpha \beta \xi} d\xi, \quad 0 \leq \xi \leq t_1
\]

Using the condition \( Q(t_1) = 0 \) in equation (5), we get

\[
q_0 = \int_0^{t_1} ((a + bt + ct^2)) e^{\alpha \beta \xi} d\xi
\]

The solution of equation (2) becomes

\[
Q(t) = a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2) + \frac{c}{3} (t_1^3 - t^3), \quad t_1 \leq t \leq T
\]

Expanding equation (6) in infinite series and integrating term by term, we have
\[ q_0 = a \sum_{n=0}^{\alpha} \alpha^n t_1^{n\beta+1} + b \sum_{n=0}^{\alpha} \alpha^n t_1^{n\beta+2} + c \sum_{n=0}^{\alpha} \alpha^n t_1^{n\beta+3} \]  

Using equation (6) in equation (5), we have

\[ Q(i) = \int_0^t (a + bt + ct^2) e^{a t^p} dt - \int_0^t (a + bt + ct^2) e^{a t^p} dt \times \frac{e^{a t^p}}{e^{a t^p}}, \quad 0 \leq t \leq t_i \]  

and

\[ = a - t_1 - t + \frac{b}{2} t_1^2 - \frac{c}{3} t_1^3, \quad t_1 \leq t \leq T \]  

The inventory level at the beginning of the cycle must be sufficient for meeting the total demand is

\[ \int_0^t (a + bt + ct^2) dt = at_1 + \frac{b}{2} t_1^2 + \frac{c}{3} t_1^3. \]  

and the total deteriorated items is

\[ q_0 - \int_0^t (a + bt + ct^2) dt = q_0 - at_1 - \frac{b}{2} t_1^2 - \frac{c}{3} t_1^3. \]  

The average inventory holding cost in \( 0, t_i \) is \( \frac{1}{2} C_q q_0 t_i \).

The average shortage cost in \( t_i, T \) is

\[ \frac{c_s}{T} \int_{t_i}^T (a + bt + ct^2) (T - t) dt = \frac{c_s}{6T} \left[ (T - t_i)^2 \{ 3a + b(T + 2t_i) + \frac{c}{3} (t_1^2 + 2t_i T) \} \right] \]  

Therefore, the total variable cost per unit time is

\[ TVC(t_i, T) = \frac{c_s}{T} \left( q_0 - at_1 - \frac{b}{2} t_1^2 - \frac{c}{3} t_1^3 \right) + \frac{1}{2} \frac{c_s}{T} q_0 t_1 + \frac{c_s}{T} \left[ c_1 (T - t_i)^2 \right], \]  

As the length of the shortage interval is a part of cycle time, therefore we may assume \( t_i = KT, \ 0 < K < 1 \); where \( K \) is a constant to be determined in an optimal manner. Using equation (7) in equation (11), we have,
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\[ TVC = \left( \frac{c_a}{T} + \frac{1}{2} c_c K_a \right) \int_0^{KT} e^{ar_0} dr + \left( \frac{c_b}{T} + \frac{1}{2} c_c K_b \right) \int_0^{KT} r T e^{ar_0} dr + \left( \frac{c_c}{T} + \frac{1}{2} c_c K_c \right) \int_0^{KT} r^2 e^{ar_0} dr \]

\[ - c_d a K - \frac{1}{2} c_d b K^2 T - \frac{1}{3} c_d c K^3 T^2 + \frac{1}{2} c_d a (1-K)^2 T + \frac{1}{6} c_d b (1-K)^3 (1+2K) T^2 \]

\[ + \frac{1}{12} c_d c (1-K)^2 (1+3K^2 T + 2KT) + \frac{c_d}{T} \]

Therefore, it can be written as,

\[ TVC = \left( \frac{1}{2} c_1 K a + \frac{a c_1}{T} \right) \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 1}{(n \beta + 1)n!} + \left( \frac{1}{2} c_1 K b + \frac{bc c_4}{T} \right) \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 2}{(n \beta + 2)n!} + \left( \frac{1}{2} c_1 K c + \frac{c c_4}{T} \right) \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 3}{(n \beta + 3)n!} - aK c_4 + \frac{1}{2} c_2 a T (1 - K)^2 - \frac{1}{2} c_2 b T K^2 + \frac{1}{6} c_2 b T^2 (1 - K)^2 (1 + 3T K^2 + 2KT) + \frac{c_3}{T} \]  

(12)

Considering \( K \) as a decision variable, the necessary conditions for the minimization of average system cost \( TVC(T,K) \) are

\[ \frac{\partial TVC}{\partial T} = 0 \quad \text{and} \quad \frac{\partial TVC}{\partial K} = 0 \]  

(13)

Then equation (14) becomes

\[ \frac{1}{2} c_2 a \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 2}{n!} + \frac{1}{2} c_1 b \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 3}{n!} \]

\[ + \frac{1}{2} c_1 c \sum_{n=0}^{\infty} \frac{a^n (KT)^n \beta + 4}{n!} + c_4 a \sum_{n=0}^{\infty} \frac{n \beta + 1}{n \beta + 1} \frac{a^n (KT)^n \beta + 1}{n!} \]

\[ + c_4 b \sum_{n=0}^{\infty} \frac{n \beta + 2}{n \beta + 2} \frac{a^n (KT)^n \beta + 2}{n!} + c_4 c \sum_{n=0}^{\infty} \frac{n \beta + 3}{n \beta + 3} \frac{a^n (KT)^n \beta + 3}{n!} \]

\[ + \frac{1}{2} c_d a T^2 (1 - K)^2 - \frac{1}{2} c_d b T^2 K^2 + \frac{1}{3} c_d b T^2 (1 - K)^2 (1 + 2K) \]

\[ + \frac{c c_4}{12} (1 - K)^2 (3K^2 T^2 + 2KT^2) - \frac{2}{3} c_d c (KT)^3 - c_3 = 0 \]  

(14)
The optimal values of $T$ and $K$ are obtained by solving equation (14) and (15). The sufficient conditions that these values minimize $TVC(T, K)$ are

\[
\frac{\partial^2 TVC}{\partial T^2} \cdot \frac{\partial^2 TVC}{\partial K^2} - \left( \frac{\partial^2 TVC}{\partial T \partial K} \right)^2 > 0
\]

(16)

and

\[
\frac{\partial^2 TVC}{\partial T^2} > 0 \quad \frac{\partial^2 TVC}{\partial K^2} > 0.
\]

Equations (14) and (15) can only be solved with the help of a computer for a given set of parameter values by truncating the infinite series if $(KT) < 1$.

4. Numerical Analysis

Equations (14) and (15) are solved with the help of a computer based technique using the following parameter values:

- $c_1 = $Rs. 100.00 per unit per day,
- $c_2 = $Rs. 10.00 per unit per day,
- $c_2 = $Rs. 20.00 per order,
- $c_4 = $Rs. 4.00 per unit,
- $\alpha = 0.002, \beta = 1.5, \alpha = 2.0, \beta = 2.0$ and $c = 5.0$

Then the optimal cycle time is $T^* = 0.443189$, optimal value $K^* = 0.9977$, economic order quantity $q^*_e = 1.50449$ units, the value of $t^*_1 = 0.0749091$ days, and total average cost $TVC^* = $Rs. 63.3948 per day. It is checked that this solution satisfies the sufficient conditions given in equation (16) and (17).

5. Sensitivity Analysis

We now study the effects of changes in the value of system parameters $a, b, c, a, \beta, c_1, c_2, c_3, c_4$ on the optimal cycle time $T^*$, the optimal length of
inventory \( q_0^* \) and the minimum total relevant cost per unit time \( TVC^* \). The sensitivity analysis is performed by changing each of the parameter by 50%, 20%, -20% and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 1.

On the basis of the results of Table-1, the following observations can be made:

1) \( T^*, q_0^* \) and \( TVC^* \) are all insensitive to changes in the parameter \( b, c \).

2) \( \alpha \) and \( \beta \) are infeasible towards the changes in \( T^*, q_0^* \) and \( TVC^* \).

3) With increase in \( c_2, c_1, c_4 \); \( q_0^* \) and \( TVC^* \) increases, but \( T^* \) decreases with increase in \( c_2 \) and \( c_4 \) and increases with increase in \( c_3 \).

4) \( c_1 \) is infeasible towards the solution. With increase in \( c_1 \); \( T^*, q_0^* \) and \( TVC^* \) are insensitive to changes in the parameter \( c_1 \).

Table-1 Sensitivity Analysis

<table>
<thead>
<tr>
<th>Changing Parameter</th>
<th>% change in the system parameter</th>
<th>% change in ( T^* )</th>
<th>% change in ( q_0^* )</th>
<th>% change in ( TVC^* )</th>
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<td>( a )</td>
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<td></td>
<td>50</td>
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<td>35.13</td>
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<td>( b )</td>
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<td>0.79</td>
<td>0.15</td>
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<td></td>
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<td>21.81</td>
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6. Conclusion

An inventory replenishment policy is developed for deteriorating items with time-quadratic demand. The rate of deterioration is time-proportional and the time to deterioration is followed by a two-parameter Weibull distribution. In the present paper, we reconsider the model of Covert and Philip (1973) and extended it to a time-dependent demand rate and shortages in inventory and also using Weibull distribution. A numerical example is taken to illustrate the theory. The sensitivity of the optimal solution to changes in the parameter values is examined. From the above analysis, it is seen that $\alpha$ and $\beta$ are the critical parameter in the sense that any error in the estimation of $\alpha$ and $\beta$ resulting errors in the optimal results. Therefore, proper care must be taken to estimate $\beta$. Again the above analysis shows that great care should be taken to estimate the value of the parameter $C_2, C_4$ and $C_4$.

References


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BIOGRAPHICAL NOTES:

Dr. Rehena Begum is currently an Assistant Professor in Assistant Professor in Mathematics in C.V.Raman College of Engineering, Bhubaneswar, Odisha, India. She got her M.Sc degree in Mathematics and M.Phil degree in Operational Research from Sambalpur University, Odissa. She obtained her PhD degree from Berhampur University. Her research interests are in the field of analysis of deteriorating inventory production system, production inventory control, optimization and cosmology. She has published articles in reputed journals like International Journal of Systems Science, British Journal of Applied Science and Technology, Journal of Scientific Research and Applied Mathematical Sciences.

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