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COMPARISON OF BAYESIAN METHOD AND CLASSICAL CHARTS IN DETECTION OF SMALL SHIFTS IN THE CONTROL CHARTS

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ABSTRACT

This paper proposes a Bayesian estimator for a quick detection of “small shifts” in the control chart under the assumption that the Process characteristic follows normality. Bayesian estimate is compared to that of the classical Moving Average, CUSUM and EWMA control charts in detection of small shifts in the process mean in the case of individual observation through a simulation study.

Control charts of Shewhart (1926) are widely used to monitor a stable process by plotting a sequence of sample data in time order on the charts. Such samples are usually taken as independent samples from the process either with fixed sample size or with fixed sampling interval.

A major disadvantage of the Shewhart control charts is that it only uses the information about the process contained in the last plotted point, and it ignores any information given by the entire sequence of points. This feature makes the Shewhart control charts relatively insensitive to small shifts in the process. However, the use of Moving Average, CUSUM and EWMA control charts are very effective alternative to the Shewhart control chart. The MACC, Montgomery (1991), is preferable in the case of individual observation which occurs very often in practice

Obviously control charts can issue a single of a shift in the process mean a substantial amount of time after the shift in the process mean actually occurred. Samuel *et.al* (1998) have suggested a Maximum Likelihood Estimator for the time of change of the process mean once the Shewhart control chart issues a signal, based on the methodology due to Hinkley (1970).

In this paper, a Bayesian estimator is derived for the detection of the time of small shifts in the mean of normal process and Bayesian estimator is compared to that of the MA, CUSUM and EWMA control charts in detection of a small shifts in the control chart.

An outline of this paper is as follows: Bayesian estimator is derived in section 2 and the MACC is briefly explained in section 3, CUSUM and EWMA are respectively explained in sections 4 and 5, while numerical example is provided in section 6. Conclusion is presented in section 7.

1. Bayesian Estimator

Suppose that $X_1, X_2, \dots, X_n; n \geq 3,$ is a sequence of independent random observations drawn from a production process and that for some unknown

$m\ (m = 1, 2, \dots, n-1),$ the distribution of the X_i's are given by

$X_1, X_2, \dots, X_m$ are i.i.d. $N(\emptyset_0, \sigma^2,)$ and

$X_{m+1}, \dots, X_n$ are i.i.d. $N(\emptyset_1, \sigma^2,)$

where $\sigma^2 > 0$ and $\emptyset_0 \neq \emptyset_1$. Then the likelihood function is

$$L(S) = (2\pi\sigma^2)^{\frac{-n}{2}} exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(X_i - \emptyset_0)^2 + \sum_{i=m+1}^{n}(X_i - \emptyset_1)^2\right]\right\}$$

$$m = 1, 2, \dots, n-1 \qquad (2.1)$$

where $S = (X_1, X_2, \dots, X_n)$ is the sequence of observed values. L is considered a function of the unknown parameters $\emptyset_0, \emptyset_1, m$ and σ^2.

We assume that $\emptyset_0 \neq \emptyset_1$ and the priori m has a uniform distribution over the possible change points, i, e.,

$$\pi_0(m) = (n-1)^{-1}; m = 1, 2, \dots, n-1 \qquad (2.2)$$

The prior distribution for the parameters are assigned as follows: The Jeffrey vague prior density for σ^2

$$\pi_0(\sigma^2) = \begin{cases} \frac{1}{\sigma^2}, & 0 < \sigma^2 < \infty \\ 0, & Otherwise \end{cases} \tag{2.3}$$

and

$$\pi_0(\emptyset_0, \emptyset_1,) = \begin{cases} \alpha, \ Constant & -\infty < \emptyset_0, \emptyset_1 < \infty \\ 0, & Otherwise \end{cases} \tag{2.4}$$

for $\emptyset_0$ and $\emptyset_1$.

The likelihood function (2.1), combined with prior distributions of $m, \sigma^2, \emptyset_0$ and $\emptyset_1$ gives the joint posterior density,

$$\pi_1(m, \sigma^2, \emptyset_0, \emptyset_1,) \propto (\sigma^2)^{-\left(\frac{n}{2}+1\right)} exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(X_i - \emptyset_0)^2 + \sum_{i=m+1}^{n}(X_i - \emptyset_1)^2\right]\right\} \tag{2.5}$$

where $m = 1, 2, \ldots, n-1;\ 0 < \sigma^2 < \infty;\ -\infty < \emptyset_0, \emptyset_1 < \infty$

Integrating (2.5) with respect to $\sigma^2, \emptyset_0$ and $\emptyset_1$, one can obtain the marginal posterior mass function for m,

$$\pi_1(m) = \begin{cases} \frac{E_{\emptyset_0}[1-g_{n-1}\ (h(m,\emptyset_0))]}{[m(n-m)]^{1/2}\ [S_1^m + S_{m+1}^n]^{\frac{n-2}{2}}}; & m = 1, 2, \ldots, n-1 \\ 0; & Otherwise \end{cases} \tag{2.6}$$

$$where,\ \ S_j{}^k = \sum_{i=j}^{k}\left(X_i - \bar{X}_j{}^k\right);\ \sum_{i=j}^{k} X_i/(k-j+1)$$

$$h(m,\emptyset_0)=\left[\frac{(n-1)(n-m)}{m(n-2)}\right]^{1/2}\frac{\left(S_1^m+S_{m+1}^n\right)^{1/2}\ \emptyset_0\ +m^{1/2}(n-2)^{1/2}(\bar{X}_1^m-\bar{X}_{m+1}^n)}{\left(S_1^m+S_{m+1}^n\right)^{1/2}\left(1+{\emptyset_0}^2/(n-2)\right)^{1/2}}$$

$g_{n-1}(X)$ is the distribution function of student's t on $(n-1)$ degrees of freedom and $E_{\emptyset_0}[f(\emptyset_0)]$ is the expectation of $f(\emptyset_0)$ taken with respect to a student's t distribution on $(n-2)$ degrees of freedom, more details can be found in Broemeling (1985).

By assuming the squared error loss function, the posterior mean of the distribution, (2.6), is taken as the Bayes estimator of m, "a change point in the process mean".

1. The Moving Average Control Charts (MACC)

The Moving Average of Span w at time "i" is defined as $M_i = X_i + X_{i-1} + \cdots + X_{i-w+1}/w$

That is, at time period i, the oldest observation in the moving average set is dropped and the newest one added to the set.

The variance of the MA, M_i is given by $V(M_i) = \frac{1}{W^2}\sum_{j=i-w+1}^{i} V(X_j)$

$$= \frac{1}{W^2}\sum_{j=i-w+1}^{i} \sigma^2 = \frac{W\sigma^2}{W} = \frac{\sigma^2}{W} \qquad \ldots (3.1)$$

Therefore, if $\emptyset_o$ denotes the target value of the mean, the control limit for M_i are

$$UCL = \emptyset_o + \frac{\sigma}{\sqrt{W}} \text{ and } LCL = \emptyset_o - \frac{\sigma}{\sqrt{W}} \qquad \ldots (3.2)$$

Plotting M_i on a control chart with UCL and LCL is given by (3.2) and concluding that the process is out of control if M_i exceeds the control limits.

4. Algorithmic CUSUM

In this section we briefly show how an algorithmic CUSUM may be constructed for monitoring the mean of a process.

Let X_i be the i^{th} observation on the process. When the process is in control, X_i has normal distribution with mean ϕ_0 and standard deviationσ. We assume that either σ is known or an estimate is available.

The tabular CUSUM works by accumulating deviations from ϕ_0 that are above target with one statistic C^+ and accumulating deviations from ϕ_0 that are below target with another statistic C^- more details can be found in Montgomery (1991). They are computed as follows:

$$C_i^+ = Max\left[0,\ X_i - (\phi_0 + k) + C_{i-1}^+\right] and\ C_i^- = Max\left[0,\ (\phi_0 - k) - X_i + C_{i-1}\right]$$

where the starting values are $C_0^+ = C_0^- = 0,\ k$ is usually called the reference value and it is often chosen about half way between the target ϕ_0 and the out of control value of the mean ϕ_1 that we are interested in detecting quickly, i.e., K = $\frac{|\phi_1 - \phi_0|}{2} = \frac{\partial\sigma}{2}$. If either $C_i^+\ or\ C_i^-$ exceeds the decision interval H, the process is considered to be out control.

5. Exponentially Weighted Moving Average Control Chart

The Exponentially Weighted Moving Average (EWMA) is also a good alternative in detecting small shifts. The performance of WEMA control chart is approximately equivalent to that of the CUSUM control chart and it is easier to setup and operate.

The exponentially moving average is defined as

$$Z_i = \lambda X_i + (1-\lambda)\ Z_{i-1}$$

where $0 < \lambda \leq 1$is a constant and the starting value is the process target, so that $Z_0 = \phi_0$ and therefore, $Z_0 = \overline{X}$

The EWMA Z_i is a weighted average of all previous sample mean than,

$$Z_i = \lambda\ X_i + (1-\lambda)\ Z_{i-1}$$

$$= \lambda X_i + (1-\lambda)\left[\lambda\ X_{i-1} + (1-\lambda)\ Z_{i-2}\right]$$

Continuing to substitute recursively for Z_{i-j}; j = 2, 3, ..., t , one can get

$$Z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j\ X_{i-j} + (1-\lambda)^i\ Z_0$$

The weights $\lambda(1-\lambda)^j$ decrease geometrically with the age of the sample and the weights sum to unity, since,

$$\lambda \sum_{j=0}^{i-1}(1-\lambda)^j = \lambda \left[\frac{1-(1-\lambda)^i}{1-(1-\lambda)}\right] = 1-(1-\lambda)^i$$

If the observations X_i are independent random variables with variance σ^2, then, the variance of Z_i is

$$\sigma_{Zi}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2i}\right]$$

The EWMA control chart would be constructed by plotting Z_i versus the sample number *i*. The limits are

$$\text{UCL} = \phi_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)\left[1-(1-\lambda)^{2i}\right]}}$$

$$\text{CL} = \phi_0$$

$$\text{LCL} = \phi_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)\left[1-(1-\lambda)^{2i}\right]}}$$

Further, $\left[1-(1-\lambda)^{2i}\right] \to 1$ as $i \to \infty$ then the control limits will approaches the following

$$\text{UCL} = \phi_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \qquad (5.1)$$

$$\text{CL} = \phi_0$$

$$\text{LCL} = \phi_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \qquad (5.2)$$

6. Numerical example

In this section we illustrate the previous results with a numerical example. We use the data from page (1995). Consider a continuous production of piston rings for an automatic engine. The inside diameter of piston rings are produced according to Normal. This was generated in an artificial sampling experiment which produced sequence of 40 observations, the first 20 having mean 5 and unit variance, and the last 20 having mean 6 and unit variance.

The posterior density given in equation (2.6) was calculated for the sequence of observation. The sequence and calculated posterior probabilities $\pi_1(m) = 1, 2, \dots, 39$ are presented in Table 1.

Table1: Sequence of observation and posterior mass function.

i	1	2	3	4	5	6	7	8	9	10
X_i	3.95	5.96	6.22	5.58	4.02	4.97	3.46	4.29	4.65	5.66
$\pi_i(i)$	0.0032	0.0008	0.0004	0.0003	0.0005	0.0006	0.0020	0.0042	0.0070	0.0047
i	11	12	13	14	15	16	17	18	19	20
X_i	5.44	5.91	4.98	3.58	5.26	3.98	4.19	6.66	6.05	5.97
$\pi_i(i)$	0.0040	0.0025	0.0033	0.0157	0.0173	0.0866	0.4782	0.0833	0.0406	0.0237
i	21	22	23	24	25	26	27	28	29	30
X_i	7.14	6.22	4.76	6.60	5.72	4.88	5.44	5.03	5.66	5.56
$\pi_i(i)$	0.0052	0.0031	0.0056	0.0026	0.0023	0.0040	0.0047	0.0083	0.0087	0.0104
i	31	32	33	34	35	36	37	38	39	40
X_i	6.37	6.66	5.10	5.80	6.29	5.49	4.93	6.18	8.29	6.34
$\pi_i(i)$	0.0057	0.0026	0.0050	0.0055	0.0037	0.0066	0.0489	0.0856	0.0026	-

Table 2 presents the MA of period three scheme. The control limits from (3.2) are 4.423 and 5.577.

Table 2: Moving Average (MA)

Period	1	2	3	4	5	6	7	8	9
MA	5.377	5.290	5.273	4.856	4.150	4.240	4.133	4.867	5.250
Period	10	11	12	13	14	15	16	17	18
MA	5.510	5.443	4.823	4.607	4.273	4.447	4.943	5.633	

Table 2 shows the MA value corresponding to the period 17 exceeds the UCL and conclude that the process is out control at that point. So the change likely occurred between periods 16 and 17.

Table 3 presents the algorithmic CUSUM scheme. The equations for $C_1^+ \; and \; C_i^-$ are

$$C_i^+ = \max \left\lfloor 0,\, X_1 - 5.5 + C_0^+ \right\rfloor and \; C_1^- = \max \left\lfloor 0,\, 4.5 - X_1 + C_0^- \right\rfloor$$

Since K = 0.5 and $\phi_0 = 5.$

Panels (a) and (b) of Table 3 summarize the remaining calculations and the quantities N^+ and N^- in Table 3 indicate the number of consecutive periods that the CUSUMs $C_1^+ \; and \; C_i^-$ have been non-zero.

Table 3: Algorithmic CUSUM

		a			**b**		
Period	**X_i**	**$X_i - 5.5$**	C_i^+	**N^+**	**4.5 - X_i**	C_i^-	**N^-**
1	3.95	-1.55	0.0	0	0.54	0.54	1
2	5.96	0.46	0.46	1	-1.46	0	0
3	6.22	0.72	1.18	2	-1.72	0	0
4	558	0.08	1.26	3	-1.08	0	0
5	4.02	-1.48	0.0	0	0.48	0.48	1

6	4.97	-0.53	0.0	0	-0.47	0	0
7	3.46	-2.04	0.0	0	1.04	1.04	1
8	4.29	-1.21	0.0	0	0.21	1.25	2
9	4.65	-0.85	0.0	0	-0.15	0	0
10	5.66	0.16	0.16	1	-1.16	0	0
11	5.44	-0.06	0.0	0	-0.94	0	0
12	5.91	0.41	0.41	1	-1.41	0	0
13	4.98	-0.52	0.0	0	-0.48	0	0
14	3.58	-1.92	0.0	0	0.92	0.92	1
15	5.26	-0.24	0.0	0	-0.76	0	0
16	3.98	-1.52	0.0	0	0.52	0.52	1
17	4.19	-1.31	0.0	0	0.31	0.83	2
18	6.66	1.16	1.16	1	-2.16	0	0
19	6.05	0.55	1.71	2	-1.55	0	0
20	5.97	0.47	2.18	3	-1.47	0	0
21	7.14	1.64	3.82	4	-2.64	0	0
22	6.22	0.72	4.54	5	-1.72	0	0
23	4.76	-0.74	0	0	-0.26	0	0
24	6.60	1.10	1.10	1	-2.10	0	0

25	5.72	0.22	1.32	2	-1.22	0	0
26	4.88	-0.62	0	0	-0.38	0	0
27	5.44	-0.06	0	0	-0.94	0	0
28	5.03	-0.47	0	0	-0.53	0	0
29	5.66	0.16	0.16	1	-1.16	0	0
30	5.56	0.06	0.22	2	-1.06	0	0
31	6.37	0.87	1.09	3	-1.87	0	0
32	6.66	1.16	2.25	4	-2.16	0	0
33	5.10	-0.40	0	0	-0.60	0	0
34	5.80	0.30	0.30	1	-1.30	0	0
35	6.29	0.79	1.09	2	-1.79	0	0
36	5.49	-0.01	0	0	-0.99	0	0
37	4.93	-0.57	0	0	-0.43	0	0
38	6.18	0.68	0.68	1	-1.68	0	0
39	8.29	2.79	3.47	2	-3.79	0	0
40	6.34	0.84	4.31	3	-1.84	0	0

The CUSUM calculations in Table 3 show that the upper side cusum at periods 22 is C_{22}^{+} = 4.54. Since this is the first period at which $C_i^+ > H = 4$, we would conclude that the process is out control at that point. The cusum also indicates when the shift probably occurred. The counter N^+ records the number of consecutive periods since the upper cusum C_i^+ rose above the value of zero, since $N^+ = 5$ at period 22, we would concluded that the process was last in control at period 22 – 5 = 17, so the shift likely occurred between periods 17 and 18.

The EWMA calculations in Table 4 shows that the Z_i values for the periods $i = 1, 2, ,...., 21$ by taking $\lambda = 0.20$, $L = 2.0$, $\phi_o = 5$ and $\sigma = 1$. The control limits from the equations (5.1) and (4.2) are UCL = 5.6667 and LCL = 4.334, and $Z_0 = \overline{X} = 5.0$

Table 4: EWMA

Period	$\mathbf{X_i}$	$\mathbf{Z_i}$
1	3.95	4.790
2	5.96	5.024
3	6.22	5.263
4	5.58	5.327
5	4.02	5.065
6	4.97	5.046
7	3.46	4.726
8	4.29	4.639
9	4.65	4.641
10	5.66	4.845
11	5.44	4.964
12	5.91	5.153

13	4.98	5.118
14	3.58	4.838
15	5.26	4.923
16	3.98	4.734
17	4.19	4.625
18	6.66	5.032
19	6.05	5.236
20	5.97	5.382
21	7.14	5.734

It is observed that Z_i value corresponding to the period 21 exceeds the UCL and conclude that the process is out of control at that point. So the change likely occurred between the periods 20 and 21.

Examination of the posterior mass function in Table 1, reveals that it peaks at $m = 17,$ whereas the true value of the time of a step change is $m = 20$. This is not surprising because of the usually high value of X_{18}, X_{19} and X_{20}. By assuming the squared error loss function, the posterior mean is $\emptyset_m = 20.62,$ in this case being closer to the true "shift point" in the process mean.

7. Conclusion

The purpose of this paper is to discuss a Bayesian method of estimating the time of shift point in the process mean. Results were obtained for the time of change in the control charts, through the Bayesian method and control chart methods and concluded that the posterior mean of the distribution is the best estimator for the quick detection of change in time of a step change in the control charts.

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