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STOCHASTIC DEGRADATION MODELS

R. Elangovan* and S. Sivanesan

Department of Statistics, Annamalai University,

Annamalai Nagar - 608 002. Tamilnadu, India

Email: sivanesantvp@gmail.com,

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ABSTRACT

Many products and systems age, wear, or degrade over time before they fail or break down. In experiments where failure times are sparse, degradation analysis is useful for the analysis of failure time distributions in reliability studies. A relationship between component failure and amount of degradation makes it possible to use degradation models and data to make inferences and predictions about a failure-time distribution. Degradation modeling techniques have generated a great amount of research in reliability field. Relative to failure-based reliability, degradation-based reliability has received a modest amount of attention in the open literature. Also, degradation analysis for reliability has attracted considerable attention of statisticians, and engineers in recent years. For instance, the failures of many manufactured products are caused by certain degradation mechanisms. In particular, unit of such a product, degradation measurements can be made over time. Such degradation measurements can be used to make inference on the lifetime distribution of the product. This motivates the need for developing stochastic degradation models using accelerating variables for inference based on both observed failure level, and degradation measurement. In this contest, stochastic degradation models, based on Shock models and cumulative damage approach, using accelerating variables to characterize the degradation phenomenon are developed.

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1. INTRODUCTION

Many products and systems age, wear, or degrade over time before they fail or break down. In experiments where failure times are sparse, degradation analysis is useful for the analysis of failure time distributions in reliability studies. Degradation measurements can be made over time. Such degradation measurements can be used to make inference on the lifetime distribution of the product. In particular, unit of such a product, degradation measurements can be made over time. Such degradation measurements can be used to make inference on the lifetime distribution of the product. Degradation analysis is especially useful for tests in which soft failures occur; that is, the lifetime of the test item is said to end after the measured performance decreases to predetermined threshold value that designates a non-functioning state or an incipient failure.

A relationship between component failure and amount of degradation makes it possible to use degradation models and data to make inferences and Predictions about a failure-time distribution. In this contest, stochastic degradation models, based on Shock models and cumulative damage approach, using accelerating variables to characterize the degradation phenomenon are developed. Degradation models can be divided into two broad categories: the flaw growth model and the flaw generation model. Traditionally, regression models are used the main probabilistic models for flaw growth. In regression models, the growth of a flaw is divided into two separate parts. First, a deterministic regression function is used to model the average growth path. The choice of the regression function is typically based on past experiences or expert judgment. In many cases, the regression function is simply chosen as a linear function of time. Parameter estimation of the regression models is usually performed using the least squares method or various maximum likelihood (ML) methods, depending on the assumed error structure of the model refer to (Weisberg, 2005). To avoid some of the inherent limitations of these classical probabilistic models, stochastic based models are later introduced as alternative modeling tools for both flaw growth and flaw generation. Unlike the classical models, the stochastic models try to imitate the flaw growth path and flaw generation directly by using a collection of random variables indexed by time. Complex covariances structures of the flaw growth and flaw generation over time and across the population can thus be established using various stochastic models that are available in current statistical literature.

2. EARLY AND RECENT DEVELOPMENTS IN STOCHASTIC DEGRADATION MODELS

Early applications of the stochastic degradation models are mostly found in the fatigue of metal and other composite materials. For example, Birnbaum and Saunders (1958) investigated the fatigue damage of structures under dynamic loads using the renewal process model. Paris et al. (1961) utilized a non-linear general path model, known as the Paris- Erdogan law, to express the fatigue crack growth over time. Lately, applications of the stochastic models are extended to degradation phenomena from a much broader range of areas, such as bridge deck deterioration refer to Madanat et al., (1995). Meeker and Escobar (1998) provide a useful summary of degradation models, emphasizing the use of linear models with assumed log normal rates of degradation. Gillen and Celina (2001). Meeker et al. (2002) discussed general approaches to estimating lifetime distributions in accelerated life tests for highly variable environments.

Accelerated degradation models for failure based on geometric Brownian motion and gamma process has been discussed by Park and Padgett (2005). A cumulative damage model for failure with several accelerating variables has been discussed by Park and Padgett (2007). Modeling zoned shock effects on stochastic degradation in dependent failure processes has been discussed by Lee et.al (2015) Accelerated burn- in and condition based on maintenance for n-subpopulations subject to stochastic degradation has been discussed by Yisha Xiang et.all. (2016).

3. STOCHASTIC DEGRADATION MODELS AND ITS APPLICATIONS

The main objective of the paper is to develop methods for the accurate estimation of Stochastic degradation models using uncertain inspection data. Four typical stochastic models are considered, namely, the random degradation rate model, Poisson process model, the gamma process model and the Weiner process model. The random rate model and the gamma process model are used to model the flaw growth, and the Weiner process model is used to model the flaw generation. Likelihood functions of the three stochastic models from noisy and incomplete inspection data are derived. Numerical examples are also provided in Section.6.

3.1 RANDOM VARIABLE MODEL

A random variable model (also referred as the general path model) is a stochastic process model that describes the flaw growth in a group of components using a deterministic function with random parameter and can be described using the following equation

$$X(t) = g(t; \theta), \qquad \cdots (3.1)$$

Where g a deterministic function of time t and random variable is θ is the parameter .The flaw growth of an individual component is then given by a deterministic function (t; θk), with its parameterθ_k a sample drawn from random variable θ. Once θ is given, the distribution of the process at any time t, $X(t)$, can be calculated using transformation techniques for functions of random variables refer to Ang and Tan. (1975).The random variable model is a very special case of the stochastic models and is widely applied to model various corrosion and wear phenomena refer to Fenyvesi et al., (2004) , Huyse and Van Roodselaar. (2010).

In this model, we mainly consider the following simple case of the random variable model called the random rate model; in which g is a linear function. Many other more complicated random variable models can be transformed into a linear random rate model using the time transform mentioned earlier. Without loss of generality, we assume the process starts from zero.

$$X(t) = R_t \qquad \cdots (3.2)$$

Equation (3.2) is used to model the flaw growth in a group of components. At time t, the degradation of the components is distributed as Rt. We call R the population flaw growth rate, or simply the population rate. On the other hand, for a specific ith component, its flaw growth path $x_i(t)$ is a deterministic function of time t: $x_i(t)$= rth, where ri is the component specific rate and is a fixed number sampled from the population rate R. It is clear that for a group of components whose degradation follows the random rate model given by equation (3.1), the mean and variance of degradation for the population are

$$E[X(t)] = tE[R], \ Var[X(t)] = t^2 Var[R].$$

The COV of the degradation is $\sqrt{Var[R]}/E[R]$ and is a constant. If inspection data are perfect, parameter estimation of the random rate model and random variable model in general, is straightforward. Since the degradation growth of each component in the population is a deterministic function of known form, its component specific parameters can be calculated from a finite number of inspection results. A statistical fitting of random variable θ using method of moments or ML estimation can then be conducted using the collection of individual growth rates. However, when large inspection uncertainties are presented, parameter estimation of the random rate model is much more complicated since in such cases the component specific rate cannot be obtained precisely.

3.2 POISSON PROCESS MODEL

In previous sections, the random rate model and the gamma process model are introduced. Both of these two models are defined on continuous sample spaces and are appropriate for modelling the flaw growth. In order to model the flaw generation, counting process models can be used, which are stochastic processes that count the number of occurrences over time. Therefore, sample space of a counting process is defined on the natural number set. Examples of counting processes include pure birth process, Bernoulli process and Poisson process. The Poisson process model is one of the most important counting processes for the Stochastic modelling of degradation. Denote the Poisson distributed random variable X with parameter $\lambda > 0$ as $X \sim Pois(\lambda)$.

Here λ is also called the Poisson rate of the distribution. The probability mass function (PMF) of X is

$$f_x(x) = f\,Pois(x;\lambda) = \frac{\lambda^x}{x!}\exp(-\lambda),\ \lambda > 0 \text{ and } x = 0,1,2,\cdots$$

The summation of two independent Poisson random variable with rate λ_1 and λ_2 is still a Poisson random variable, with rate $\lambda_1 + \lambda_2$. In addition, for the Poisson distribution, the converse of the above property, commonly known as Raikov's theorem, also holds, which is stated as: if the sum of two independent random variables is Poisson distributed, so is each of these two independent random variables is also based on the Poisson distribution. refer to Gupta et.al.,(2010). a continuous time stochastic process, $X(t), t \geq 0$, is called a homogeneous Poisson process with rate λ, if it has the following properties at time 0, X(0)=0, for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, random variables

$X(t_1) - X(0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent.

The number of occurrences between t and $t + \tau$, $X(t+\tau) - X(t)$, is Poisson distributed with rateλ_τ, for any $t \geq 0$ and$\tau > 0$, i.e.

$$X(t+\tau) - X(t) \sim Pois(\lambda_\tau).$$

From the definition, Poisson process is a stochastic process with stationary and independent increments. The mean and variance of Poisson process are

$$E[X(t)] = \lambda t,\ Var[X(t)] = \lambda t.$$

The COV of the process is $COV[X(t)] = \frac{1}{\sqrt{\lambda t}}$

obviously, parameter λ is simply the average number of occurrences per unit time.

3.3 AKIKE INFERMATION CRITERION (AIC)

The akaike information criterion is a measure of the relative goodness of fit of a statistical model. AIC values provide a means for model selection. In the general case, the AIC is

AIC = 2k – 2ln(L).

Where k is the number of parameters in the statistical model. And L is the maximized value of the likelihood function for the estimated model. For a detailed study, refer to Akaike (1974).

3.4 BAYESIAN INFORMATION CRITERION (BIC)

The Bayesian information criterion (BIC) or Schwarz criterion (also SBC, SBIC) is a criterion for model selection among finite set of models. It is based, in part on the likelihood function. And it is closely related to Akaike information criterin (AIC). When fitting models it is possible to increase the likelihood by adding parameters. But doing so may result in overfitting.

3.5 GAMMA PROCESS MODEL

The gamma process model is a stochastic process model which belongs to a broader category of degradation model called the cumulative damage model. The basic assumption of the cumulative damage model is that the degradation of a component is caused by a series of independent but random small damages. Suppose $X(t)$ is the total growth of a flaw within time interval $[0, t], [0, t]$ is discretized into k sub-intervals as $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$. Denote the flaw growth within each sub-interval as $X_t, i = 1, 2, \cdots, k$. In the cumulative damage model, Xi are regarded as independent random variables, and the total flaw growth $X(t)$ is the sum of all X_t, i.e.,

$$X(t) = X_1 + X_2 + \cdots + X_k \quad \cdots (3.3)$$

If x denote the gamma distributed random variable with shape parameter a>0 and scale Parameter $b > 0$ as $X \sim Ga(a, b)$.

The PDF of X is

$$fX(x) = f_{Ga}(x; a, b) = \frac{(x/b)^{a-1}}{b\sqrt{(a)}} \exp(-x/b), \text{for } x > 0,$$

Where $\sqrt{(a)} = \int_0^\infty t^{a-1} e^{-t} dt$ is a gamma function.

An important property of the gamma distribution is that the sum of two independent gamma random variables with the same scale parameter b is still gamma distributed. Suppose X_1 and X_1 are two independent gamma random variables, $X_1 \sim Ga(a_1, b)$ and $X_2 \sim Ga(a_2, b)$The sum of X_1 and X_2 is then a gamma random variable with shape parameter $a_1 + a_2$ and scale parameterb, i.e., $X_1 + X_2 \sim Ga(a_1 + a_2 b)$. Therefore, if X_i, $i = 1, 2, \cdots, k$, in equation (3.1) are all Gammas distributed with the same scale parameter, the total degradation $X(t)$ is also gamma distributed and its parameters can be obtained directly without conducting the time-consuming convolutions. Utilizing the above property of the gamma distribution, the gamma process model is defined as follows, a continuous-time stochastic process $\{X(t); t \geq 0\}$ with sample space$[0, +\infty)$ is called a gamma process with shape parameter α and scale parameter β $(\alpha, \beta > 0)$ is refer to Singpurwalla. (1997).

Using the properties of the stochastic process with stationary and independent increments, the mean and variance of the gamma process are given as

$E[X(t)] = \alpha\beta t, \; Var[X(t)] = \alpha\beta^2 t.$

The coefficient of variation of X (t) is

$$COV[X(t)] = \frac{E[X(t)]}{Var^{-1/2}[X(t)]} = \frac{1}{(\sqrt{\sqrt{\alpha t}})}.$$

If $\mu = \alpha\beta$ and $v = 1/\sqrt{\alpha}$, one has $E[X(t)] = \mu t, COV[X(t)] = v/\sqrt{t}$. μ is called the average rate of the gamma process model and v is the COV. μ and v are sometimes used as an alternative set of the model parameters. The PDF of X(t) in terms of μ and v is given as

$$f_{X(t)} = fGa\left(x; \frac{t}{v^2}, \mu v^2\right)$$

$$= \frac{\left[{}^{x}/_{(\mu v^2)}\right]^{t/v^2-1}}{\mu v^2\sqrt{t/v^2}} \exp\left[{}^{-x}/_{(\mu v^2)}\right]$$

.

Compared to the random rate model, the gamma process model is qualitatively different in the following two aspects:

(1) In the random rate model, the flaw growth rate for a specific component; while in the gamma process model the flaw growth is modelled as the sum of a sequence of small independent random damages and thus the rate changes continually over time;

(2) In the random rate model, flaw growth rates for different components vary across the population; whereas in the gamma process model, the future flaw growths of different component follow same distributions, regardless of their current flaw sizes. In short, random rate model and gamma process model are two extremes. The former one tries to capture the sample differences across the population while assuming there are no temporary uncertainties. And the latter one model the temporary uncertainties well but assumes the population is homogeneous in terms of future flaw growth distribution.

3.5.1 LIFETIME PREDICTION MODEL

Gamma process is a stochastic process which is applicable to modeling the always positive and strictly increasing degradation data. In mathematics, Gamma process $\{Y(t): Y(0) = 0\}$ has independent, non-negative increments $\Delta Y(t) = Y(t + \Delta t) - Y(t)$ that follow a Gamma distribution as

$$\Delta Y(t) \sim Ga\big(x\big(\Lambda(t + \Delta t) - \Lambda(t)\big).\beta\big) \qquad \cdots (3.4)$$

Where $\beta(\beta > 0)$ is a scale parameter, $x(x>0)$ is a shape parameter and where $\Lambda(t)$ is a monotone increasing function of time t with $\Lambda(0) = 0$. According to the additivity of a Gamma distribution, it can be deduced that Y(t) should follow the Gamma distribution $Ga(x\big(\Lambda(t)\big).\beta)$. The probability density function of Y(t) is expressed as

$$f(y) = \frac{\beta^{\alpha\Lambda(t)}}{\Gamma\alpha\Lambda(t)} Y^{\alpha\Lambda(t)-1} e^{-Y\beta} \qquad \cdots (3.5)$$

3.5.2 PARAMETER ESTIMATION

Suppose that a constant stress ADT was carried out. If T0 denote the normal use temperature level, Tk denote the kth accelerated temperature level, yijk denote the ith observed degradation data of the j^th sample at Tk, tijk denote the observing time, $\Delta y_{ijk} = y_{ijk} - y_{(i-1)jk}$ denote the degradation increment, where i=1,2,3….n1, j=1,2,3….n2, k=1,2,3….n3. According to equ. (3.4), the maximum likelihood function for each sample can be set up as

$$L\left(\alpha_{jk}, c_{jk}, \beta_{jk}\right) = \prod_{i=1}^{n_1} \frac{\beta_{jk}^{\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)}}{\left(\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)\right)} \exp\left(-\Delta y_{ijk}\beta_{jk}\right) \Delta_{y_{ijk}}^{\alpha_{jk}\left(t_{ijk}^{c_{jk}} - t_{(i-1)jk}^{c_{jk}}\right)-1} \quad \ldots (3.6)$$

Through equ. (3.6), the maximum likelihood estimates $\left(\widehat{\alpha}_{jk}, \hat{c}_{jk}, \widehat{\beta}_{jk}\right)$of all samples can be obtained, then the random effects of □ and □ can be evaluated. To predict the product lifetime at T0, the maximum likelihood function synthesizing all the degradation data is expressed as

$$L\left(\gamma_1, \gamma_2, \gamma_3, \gamma_4\right) = \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{k=1}^{n_3} \frac{\gamma_3^{\exp\left(\gamma_1 - \gamma_2/T_k\right)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)}}{\exp\left(\gamma_1 - \gamma_2/T_k\right)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)} \Delta_{y_{ijk}}^{\exp\left(\gamma_1 - \gamma_2/T_k\right)\left(t_{ijk}^{\gamma_4} - t_{(i-1)jk}^{\gamma_4}\right)-1} \exp\left(-\Delta y_{ijk}\gamma_3\right) \quad ..(3.7)$$

Through equ.(3.7), the maximum likelihood estimates $\left(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4\right)$can be obtained. $\left(\widehat{\alpha}_0, \hat{c}_0, \widehat{\beta}_0\right)$ at T0 can be obtained, so the lifetime of products at T0 can be evaluated.

4. DEGRADATION MODEL BASED ON WIENER PROCESS

Due to the good mathematical properties and physical interpretations of Wiener process, it has been taken to describe the performance degradation of products. A well adopted form of Wiener process $\{X(t), t \geq 0\}$ can be expressed as M_1

$$X(t) = \mu t + \sigma B(t) \quad \ldots (4.1)$$

Where μ and σ are drift and diffusion parameters, respectively; B(t) is a standard Brownian motion which is used to describe time-correlated structure. It is assumed that the degradation path is described by the modelM_1. Given the threshold valueξ, the product's lifetime T is defined as

$$T = \inf\{t | X(t) \geq \xi\} \quad \ldots (4.2)$$

and it is known that T follows inverse Gaussian distribution with probability distribution function (PDF)

$$f_T(t|\, \mu, \sigma) = \frac{\xi}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(\frac{(\xi - \mu t)^2}{2\sigma^2 t}\right)$$

$$= \sqrt{\frac{\xi^2\vartheta}{2\pi t^3}}\exp\left(\frac{\vartheta(\xi - \mu t)^2}{2t}\right) \quad \text{... (4.3)}$$

Then, basis on the PDF of lifetimeT, we can obtain the reliability at time t as follow

$$R(t) = \Pr\big(T > t\big) = \int_t^{+\infty} f_T(x)\, dx$$

$$= \Phi\left(\frac{\mu t - \xi}{\sigma^2 t}\right) - \exp\left(\frac{2\mu\xi}{\sigma^2}\right)\Phi\left(\frac{\mu t + \xi}{\sigma^2 t}\right) \quad \text{... (4.4)}$$

Where $\Phi(.)$ is the cumulative distribution function (CDF) of standard normal distribution.

In most cases, each sample unit usually experiences different sources of variations during their operation. Thus, it is more appropriate to incorporate unit to unit variability in the degradation process, and the mixed effect model can describe the unit variability. A conventional mixed effect Wiener process model can be expressed as M_2

$$\begin{cases} Y(t) = \mu t + \sigma B(t) \\ \quad \mu \sim N\eta, \sigma_\eta^2 \end{cases} \quad \text{… (4.5)}$$

Assume that the degradation path of product is described by the model M2. Considering that the drift parameter μ is random variable, by using the total law of probability, the PDF of the lifetime T can be reconstructed in model M2 as

$$f_T\left(t/\vartheta = \int_{-\infty}^{+\infty} f_T(t/\mu, \sigma)\varphi\left(\frac{\mu - \eta}{\sigma_\eta}\right) d\mu\right)$$

$$\sqrt{\frac{\xi^2}{2\pi\big(\vartheta^{-1} + \sigma_\eta^2 t\big)t^2}}\exp\left(-\frac{(\xi - \eta t)^2}{2\big(\vartheta^{-1}t + \sigma_\eta^2 t^2\big)}\right) \quad \text{... (4.6)}$$

Where φ (□) is distribution function of the standard normal distribution. Then, the reliability at time t can be expressed as

$$R(t) = \Phi\left(-\frac{(\eta t - \xi)}{\sqrt{\sigma_\eta^2 t^2 + \vartheta^{-1} t}}\right) - \exp\left(\frac{2\eta\vartheta^{-1}\xi + 2\sigma_\eta^2\xi^2}{\vartheta^{-2}}\right) \times$$

$$\Phi\left(\frac{2\sigma_\eta^2\xi t + \vartheta^{-1}(\eta t - \xi)}{\vartheta^{-1}\sqrt{\sigma_\eta^2 t^2 + \vartheta^{-1} t}}\right) \quad \text{... (4.7)}$$

A random effect Wiener process is used to Characterize the degradation data, where μ and σ of this model are regarded as random variables. A random effect Wiener Process model can be expressed as M_3

$$\begin{cases} Z(t) = \mu t + \sigma B(t) \\ \vartheta = \sigma^{-2} \sim G(\beta, \alpha) \\ \mu / \vartheta \sim N\left(\theta, \lambda/\vartheta\right) \end{cases} \quad \cdots (4.8)$$

It is noted that model M3 can be used to describe both the variation from unit to unit and time correlated structure. Similarly to the above, when μ and σ are random variables,

by using the total law of probability, the PDF of lifetime T in model M3 is given by

$$f_{T(t)} = \frac{\xi \alpha^{\beta}}{2\pi\sqrt{\lambda t^{3\Gamma(\beta)}}} \int_{-\infty}^{+\infty} \left\{ \vartheta^{\beta-1} \exp\left[-\left(\frac{(\xi-\theta t)^2}{2t(1+\lambda t)} + \alpha \right) \vartheta \int_{-\infty}^{+\infty} \exp\left[-\frac{\left(\mu - \frac{\xi\lambda+\theta}{1+\lambda t}\right)^2}{\frac{2\lambda}{(1+\lambda t)\vartheta}} \right] d\mu \right\} \right] d\vartheta$$

$$= \frac{\Gamma\left(\beta + \frac{1}{2}\right)\xi}{\sqrt{2\pi t^{3[\alpha(\lambda t+1)]\Gamma(\beta)}}} \left(1 + \frac{(\xi - \theta t)^2}{2\alpha(\lambda t^2 + t)}\right)^{-\beta-\frac{1}{2}} \quad \cdots (4.9)$$

and the reliability at time t can be expressed as

$$R(t) = 1 - \int_0^t \frac{\Gamma\left(\beta + \frac{1}{2}\right)\xi}{2\pi t^{3[\alpha(\lambda t+1)]\Gamma(\beta)}} \times \left(1 + \frac{(\xi - \theta t)^2}{2\alpha(\lambda t^2 + t)}\right)^{-\beta-\frac{1}{2}} dx \quad \cdots (4.10$$

5. DEDUCTION PROCESS FOR TEMPERATURE STRESS:

According to the acceleration factor constant hypothesis, the following equation should always be identical

$$F_k(t_k) = F_h\left(A_{k,h} t_k\right) \quad \cdots (5.1)$$

For a wiener process, its first passage times follow an inverse Gaussian distribution when a threshold is specified. The CDF of the inverse Gaussian distribution is too complex to compute, so the PDF was used as

$$f(t) = \frac{l}{\sqrt{2\pi\sigma^2 t^3}} \exp\left[\frac{(l - \mu t)^2}{2\sigma^2 t}\right] \quad \cdots (5.2)$$

From equ. (5.1), the following identical equation can be obtained

$$f_k(t_k) = A_{k,h} f_h(t_h) \quad \cdots (5.3)$$

The deduction process is illustrated as

$$f_k(t_k) = \frac{dF_k(t_k)}{dt_k} = A_{k,h} \frac{dF_k\left(A_{k,h} t_k\right)}{d\left(A_{k,h} t_k\right)} = A_{k,h} \frac{dF_h(t_h)}{d(t_h)}$$

$$= A_{k,h} f_h(t_h) \quad \cdots (5.4)$$

Substitute Eq. (5.2) into Eq. (5.3), then

$$A_{k,h} = \frac{f_k(t_k)}{f_h(t_h)} = \frac{f_k(t_k)}{f_h(A_{k,h}t_k)} = \frac{\sigma_h A_{k,h}^{3/2}}{\sigma_k} \times$$

$$\exp\left[\left(\frac{l\mu_h}{\sigma_h^2} - \frac{l\mu_k}{\sigma_k^2}\right) + \frac{1}{t_k}\left(\frac{l^2}{2\sigma_h^2 A_{k,h}} - \frac{l^2}{2\sigma_h^2}\right) + t_k\left(\frac{\mu_h^2 A_{k.h}}{2\sigma_h^2} - \frac{\mu_k^2}{2\sigma_k^2}\right)\right] \quad \cdots (5.5)$$

To ensure $A_{k,h}$ is a constant which does not change with t_k, the following relationship must be satisfied

$$\begin{cases} \sigma_k^2 = \sigma_h^2 A_{k,h} \\ \sigma_h^2 \mu_k^2 = \sigma_k^2 \mu_h^2 A_{k,h} \end{cases} \quad \cdots (5.6)$$

So the following relationship can be deduced

$$A_{k,h} = {}^{\mu_k}/_{\mu_h} = {}^{\sigma_k^2}/_{\sigma_h^2} \quad \cdots (5.7)$$

It can be concluded that both l and r should change with accelerated stresses varying and the ratios of $\mu_k \mu_h$ should be equal to that of σ_k^2, σ_h^2.

6. SIMULATION STUDY

Following the approach of Meeker and Escobar on degradation data of carbon film registers as suggested by Hao Wei et.al. (2015)., a case study was observed in the micro electro Mechanical System Lab (MEMS LAB), Faculty of Engineering and technology, Annamalai University, a total of 30 resistors in a constant stress ADT, where 9 samples were observed at 800c, 10 samples observed at 1200c and the rest 10 samples were observed at 1600c. All the samples were observed every time at t0 = 0, t1 = 0.462, t2 = 1.02, t3 = 3.841 and t4 = 6.084 (103 h). it was assume that the normal use temperature was 800c and the threshold value for percent increase in resistance was 1=5. The samples are numbered and shown in the following table for the poisson process and gamma degradation process, The results were shown Table 6.1, 6.2 and the Gamma process is also compared with Weiner process and Shown in Table 6.3.

For a more detailed discussion refer to Park and Padgett (2005), Paulsson (2005), Kharoufeh and Cox (2007), Gebraeel and Pen (2008), Goojian et.al (2009), Balakrishnan et.al. (2010), Makagawa (2011), Nakagawa(2011), Donglin (2012), Penda et al. (2013), Chouzenoux et.al. (2015), Jedark and Marcinek (2016), Kahle et.al.(2016), the Maximum Likelihood Estimates , AIC and BIC of each sample for the Stochastic degradation model using Poisson process are shown in the following table.

TABLE: 6. 1

THE MAXIMUM LIKELIHOOD ESTIMATES, AIC AND BIC OF EACH SAMPLE FOR THE STOCHASTIC DEGRADATION MODEL USING POISSON PROCESS

Temperature(°c)	Item	Poisson Process With $\lambda(t)$		
		$\hat{\lambda}$	AIC	BIC
	1	0.546	1.291	2.638
	2	0.359	-1.743	-3.543
	3	0.498	3.798	-5.643
	4	0.408	0.543	2.629
	5	0.737	-2.356	-3.537
80^0c	6	0.504	1.513	3.659
	7	0.567	2.668	3.238
	8	0.594	0.264	1.799
	9	0.423	4.434	5.003
	10	0.741	2.632	2.562
	11	0.961	3.572	5.897
	12	1.508	10.791	13.461
	13	1.134	6.113	7.864
	14	1.229	5.849	8.428
120^0c	15	1.486	7.149	9.276
	16	1.294	6.334	8.635
	17	1.084	9.378	12.463
	18	1.735	10.182	15.973
	19	1.062	7.981	9.134
	20	1.598	6.897	8.245
	21	2.793	10.352	13.246
	22	1.791	14.256	17.865
	23	1.942	19.634	23.542
	24	2.324	11.141	13.752
160^0c	25	1.468	16.398	19.283
	26	1.119	17.982	20.329
	27	1.648	11.785	14.970
	28	1.669	12.754	15.975
	29	1.728	12.879	14.895
	30	1.541	14.459	16.902

Table – 6. 2:

The Maximum Likelihood Estimates and AIC of each Sample

Temperature(^{o}c)	Item	Gamma Process with $\Lambda_1(t)$				Gamma Process with $\Lambda_2(t)$		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	AIC	$\hat{\alpha}$	$\hat{\beta}$	AIC
80^{o}c	1	5.788	16.564	0.989	-2.689	0.672	7.097	1.256
	2	6.896	25.675	0.806	-6.870	0.236	7.908	-1.643
	3	7.937	15.908	0.340	-1.547	0.852	4.675	3.785
	4	6.548	20.249	0.890	-4.097	0.265	6.834	0.534
	5	7.453	25.090	0.135	-6.239	0.564	10.654	-2.591
	6	23.325	70.785	0.097	-8.095	0.230	7.540	1.653
	7	19.437	45.457	0.706	-5.453	0.434	6.342	2.681
	8	6.572	21.587	0.648	-3.672	0.032	8.675	0.650
	9	4.543	6.096	0.905	2.975	0.347	4.980	1.980
	10	3.652	4.459	0.750	-3.451	0.973	2.657	4.460
120^{o}c	11	16.997	34.989	0.749	3.090	0.967	7.865	3.571
	12	23.678	20.575	0.127	2.092	1.546	3.543	10.123
	13	10.023	14.895	0.709	-2.978	1.832	5.897	6.787
	14	22.542	36.957	0.450	-1.334	1.098	6.882	5.221
	15	14.473	18.983	0.758	3.650	1.549	5.565	7.099
	16	9.095	14.346	0.804	2.650	1.549	5.432	6.433
	17	14.867	14.987	0.809	2.096	1.753	3.875	9.882
	18	12.387	11.001	0.658	6.905	1.902	3.241	10.344
	19	17.853	21.763	0.708	1.334	1.867	4.543	7.789
	20	18.867	26.980	0.345	1.809	1.902	4.231	6.097
173^{o}c	21	19.543	15.798	0.569	6.980	2.092	6.980	10.454
	22	68.767	38.983	0.659	3.421	1.033	5.126	14.332
	23	24.675	6.832	0.324	13.678	1.893	2.822	19.554
	24	17.475	1.781	0.432	6.985	2.943	1.452	11.532
	25	14.675	3.097	0.043	11.932	1.985	4.582	16.990
	26	3.692	25.564	0.980	16.098	1.467	1.546	17.762
	27	34.248	20.675	0.902	14.642	1.945	1.092	12.775
	28	40.651	27.976	0.029	3.543	1.435	3.541	11.560
	29	36.237	25.098	0.032	4.087	1.762	3.780	10.432
	30	40.189	2o.098	0.980	4.421	1.369	3.342	12.980

It was assumed that the normal use temperature was 80o C and the threshold value for percent increase in resistance was l = 5. The sample numbered 27 was omitted in our example, because its degradation data was not strictly increasing and the Gamma process was expected to model the degradation data. The Degradation of resistors can be seen that the resistances of the samples at three different stress levels uniformly show a sudden augment at the beginning of the ADT, and then reach a stably increasing process.

Thus, when modeling the degradation data with a Gamma process, we specified $\Lambda_1(t) = t^c, \Lambda_2(t) = t$, respectively. The maximum likelihood estimates and Akaike information criterion (AIC) of each sample were listed in Table6.3. It can be concluded that a Gamma process with $\Lambda_1(t)$ is more suitable for modeling the degradation data of the resistors by comparing the AICs in Table 6.2. Through Eq. (3.7), $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4$ were obtained and listed in Table 6.3. For a compare purpose, a wiener process was also used to model the accelerated degradation data. Through the acceleration factor constant hypothesis, the relationships between parameters of a Wiener process and temperature stress was deduced as $\mu = \gamma_1 - \gamma_2/T$ and $\sigma = \exp\left(0.5\left(\gamma_1 - \gamma_2/T\right)\right)$. Where μ drift parameter, σ denote the diffusion parameter and $\gamma_1, \gamma_2, \gamma_3$ are coefficients. The Gamma process was more suitable to modeling the accelerated degradation data than the wiener process, since the AIC of the gamma process was smaller. The estimated $\hat{\alpha}_0$ at 50o C is 0.534. and mean time to failure (MTTF) of resistors at 80o C can be evaluated as $\hat{\xi}_{BS} = 5.991 \times 10^6$h .

Table – 6.3:

The Maximum Likelihood Estimates and AICs Obtained by modeling All accelerated degradation data.

Model	Parameter				AIC
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	
Gamma	10.654	364.865	6.721	0.234	9.706
Weiner	10.086	453.231	9.321	0.645	45.762

7. CONCLUSION

Probabilistic modelling of degradation is important to reliable and efficient maintenance of large engineering systems. Traditionally, probabilistic modelling of degradation are carried out using regression analysis and extreme value analysis. Although effective in many cases, these traditional approaches have their inherent limitations. Stochastic based models are therefore used as alternative modelling tools for some degradation phenomena. This Paper introduces three common stochastic models, the random rate model, poisson process model, the gamma process model and the Weiner process model. The random rate model and the gamma process model are suitable for modelling flaw growth, and the Weiner process model are usually used for modelling flaw generation. Likelihood functions of the models are also derived under the assumption that the inspection data are accurate. The derived likelihood function can then be used to estimate the model parameters when there are little or no inspection uncertainties. However, if large inspection uncertainties are presented, likelihood function considering the inspection uncertainties should be used. The acceleration factor constant hypothesis provides an appealing approach to deduce the relationship between parameters of a degradation model and accelerated stresses. Although the deduced conclusions still need abundant test data to verify, the hypothesis and deducing method are more convincing than the assumptions made from experience.

REFERENCES

1. Akaike, H. (1974). A New Look at the Statistical Model, IEEE Transactions on Automatic Control, 19(6), pp.716-723.
2. Birnbaum, Z. W. and S. C. Saunders, (1969.) A new family of life distributions, Journal of Applied Probability, Vol. 6, pp. 319–327.
3. Lawless. J and Martin. C. (2004). Covariates and random effects in a Gamma process model with application to degradation and failure. Lifetime Data Analysis, 10(3),pp. 213-219.
4. Lu C.J, Meeker W.Q (1993). Using degradation measures to estimate a time-to failure distribution. Technometrics 35(2), pp. 161–174.
5. Meeker, W.Q and Escobar, A. (1998).Statistical methods for reliability data. New York: John Wiley & Sons, pp. 638–48.
6. Meeker, W.Q. and Hamada, M. (1995). Statistical tools for the rapid development and evaluation of high-reliability products. IEEE Transactions on Reliability, 44(2), pp. 187– 198.
7. Park, C. and W. J. Padgett, (2005). New cumulative damage models for failure using Geometric Brownian motion and gamma processes. Lifetime Data Analysis, 11(4),pp. 511–527.
8. Tang, L.C. and Shang, C.D. (1995). Reliability prediction using nondestructive accelerated-degradation data: case study on power supplies. IEEE Transactions on Reliability 44(4), pp.562-566.
9. Meeker, W.Q. and Hamada, M. (1995) Statistical tools for the rapid development and evaluation of high-reliability products. IEEE Transactions on Reliability, 44(2),pp.187– 198.

10. Xiang, Y., Coit, D.W. and Feng, Q. (2013). n Subpopulations experiencing stochastic degradation: reliability modeling, burn-in, and preventive replacement optimization. IIE Transactions, 45(4), pp.391–408.
11. Yang. K and Yang. G (1998). Degradation reliability assessment using severe critical values. International Journal of Reliability, Quality and Safety Engineering, 5(1),pp. 85-95.
12. Ye Z. S, Wang Y, Tsui K.L (2013). Degradation data analysis using Wiener processes with measurement errors. IEEE Transaction on Reliability; 62(4):pp.772–780.